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Numerical Solution of the One-Pair Equation in the Massive Schwinger Model

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We describe various numerical methods for the solution of the one-pair light-cone equation in the massive Schwinger model. Results obtained by these methods, and by others, are compared. Possible extensions to the coupled set of one-pair and two-pair equations are discussed. © 1989 Academic Press, Inc.

I. INTRODUCTION

The massive Schwinger model [1] is frequently cited [2] as an example of a theory in which the elementary fermions are confined. A linear potential between fermions is obtained directly from field theory and is a natural result of having only one spatial degree of freedom. This property and the reduced number of dimensions make study of the Schwinger model ideal as a first step in any attempt to understand confining theories.

A field-theoretic statement of the mass-eigenvalue problem can be converted to an infinite set of coupled integral equations [3]. This is done by expanding the mass eigenstate in a sum over Fock states. The expansion works best in theories quantized on the light cone [4, 5].

To obtain a manageable set of equations, the Fock-state expansion is truncated. For a meson-like state, the most severe truncation leaves only the fermion–anti-fermion Fock state. The associated integral equation is the one-pair light-cone equation [6, 7]

$$\left(m^2-\frac{g^2}{\pi}\right)\sum_{i=1}^{2}\frac{1}{x_i}\psi(x_1,x_2)+\frac{g^2}{\pi}\int_0^1 dx_1'\psi(x_1',x_2') + \frac{g^2}{\pi}\mathscr{P}\int_0^1\frac{dx_1'}{x_1-x_1'}\frac{d}{dx_1'}\psi(x_1',x_2')=M^2\psi(x_1,x_2). \tag{1.1}$$

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Here m is the fermion mass, g the coupling constant, and M the bound-state mass. The $\mathscr{P}$ in the third term indicates that the principal part of the integral should be used. The x_i are momentum fractions, which are ratios of the light cone plus component of the fermion momenta p_i to the plus component of the total momentum p:

$$x_i = \frac{p_i^+}{p^+}, \qquad p_i^+ \equiv p_i^0 + p_i^z, \qquad p^+ \equiv p^0 + p^z. \tag{1.2}$$

They vary between zero and one. Momentum conservation requires that they satisfy

$$x_1 = 1 - x_2 \equiv x. \tag{1.3}$$

Each term of (1.1) has a direct interpretation. The first is the kinetic energy contribution; it includes a finite mass renormalization. The second term in (1.1) comes from pair annihilation. The third is the Fourier transform of the linear potential.

Appropriate boundary conditions follow from requiring a finite expectation value for the kinetic energy. They are

$$\psi(x=0) = \psi(x=1) = 0. \tag{1.4}$$

In the next section we consider various numerical methods [7] for the solution of (1.1). Following Hanson *et al.* [8], we will work with a dimensionless form

$$\frac{2}{\pi}\frac{R-2}{x(1-x)}\psi(x) + \frac{4}{\pi}\int_0^1 dx'\psi(x') + \frac{4}{\pi}\mathscr{P}\int_0^1 \frac{dx'}{x-x'}\frac{d}{dx'}\psi(x') = \mathscr{E}\psi(x), \tag{1.5}$$

where

$$R = 2\pi\frac{m^2}{g^2}, \qquad \mathscr{E} = 4\frac{M^2}{g^2}. \tag{1.6}$$

Each method approximates this integral equation by a finite matrix eigenvalue problem. The matrix problem is solved by standard means.

The results from each method are compared in Section III. Possible extensions to the coupled set of one-pair and two-pair equations are also discussed. Results from the best method for that purpose are compared with eigenvalues obtained with the method of Eller, Pauli, and Brodsky [9]. They carry out discretization at the field-theory level, which is very different from the approach taken here.

II. Numerical Methods

There are many methods that can be applied to the eigenvalue problem (1.5). One such, the Multhopp method, has already been applied [8] to a sightly

different equation that arises in the large-N limit of SU(N) Yang–Mills. In Section II.A we adapt the analysis to the present case. The second method considered is based on Gauss–Chebyshev quadrature for singular integrals [10]; it is discussed in Section II.B. Two finite-element methods [11] are presented in Section III.C. One uses linear elements and the other, quadratic. A method not discussed here is the variational approach of Bergknoff [12].

A. *Multhopp*

The basis of the Multhopp approach [8] is an approximate sine-function expansion for the wave function, combined with a judicious choice of collocation points. A new independent variable is introduced, defined implicitly by

$$x = \tfrac{1}{2}(1 - \cos\theta). \tag{2.1}$$

The wave function is then approximated by

$$\psi \approx \sum_{j=1}^{N} a_j \sin j\theta. \tag{2.2}$$

Note that the boundary conditions (1.4) are automatically satisfied. On substitution of (2.2), Eq. (1.5) becomes

$$\frac{8}{\pi}\frac{R-2}{\sin^2\theta}\psi(\theta) + \frac{2}{\pi}\sum_{j=1}^{N} a_j \int_0^{\pi} \sin\theta \sin j\theta \, d\theta + \frac{16}{\pi}\sum_{j=1}^{N} a_j j \mathscr{P}\int_0^{2\pi} \frac{\cos j\theta' \, d\theta'}{\cos\theta' - \cos\theta} = \mathscr{E}\psi(\theta). \tag{2.3}$$

The first integral is trivial. The second is most easily done via contour integration around the unit circle. The principal part is taken in the usual way. The result is

$$\mathscr{P}\int_0^{2\pi} \frac{\cos j\theta' \, d\theta'}{\cos\theta' - \cos\theta} = \pi \frac{\sin j\theta}{\sin\theta}. \tag{2.4}$$

Equation (2.3) can then be written as

$$\frac{8}{\pi}\frac{R-2}{\sin^2\theta}\psi(\theta) + \sum_{j=1}^{N} a_j \delta_{j1} + 8\sum_{j=1}^{N} a_j \frac{j\sin j\theta}{\sin\theta} = \mathscr{E}\psi(\theta). \tag{2.5}$$

To have the necessary number of linear equations, N collocation points are specified. The most useful ones are the Multhopp angles [8]

$$\theta_k = \frac{k\pi}{N+1}, \qquad k = 1, \ldots, N. \tag{2.6}$$

Their usefulness follows from the completeness relation

$$\sum_{l=1}^{N} \sin l\theta_j \sin l\theta_k = \frac{N+1}{2}\delta_{jk}. \tag{2.7}$$

With this identity, one can eliminate a_j from (2.5) in favor of $\psi(\theta_k)$. To invert (2.2) at θ_k, simply multiply by $\sin j\theta_k$ and sum over k. Use of the completeness relation then yields

$$a_j = \frac{2}{N+1}\sum_{k=1}^{N} \psi(\theta_k) \sin j\theta_k. \tag{2.8}$$

From (2.5) and (2.8), we finally obtain a matrix eigenvalue problem

$$\sum_{k=1}^{N} A_{mk}\psi(\theta_k) = \mathscr{E}\psi(\theta_m) \tag{2.9}$$

with

$$A_{mk} = \frac{8}{\pi}\frac{R-2}{\sin^2\theta_m}\delta_{mk} + \frac{2}{N+1}\sum_{j=1}^{N} \sin j\theta_k \left[\frac{8j \sin j\theta_m}{\sin\theta_m} + \delta_{1_j}\right]. \tag{2.10}$$

The matrix problem can be solved by standard means.

A reduction in computational effort can be achieved if symmetries are taken into account. By a redefinition of the eigenvector components

$$\tilde{\psi}(\theta_k) = \sqrt{\sin\theta_k}\,\psi(\theta_k) \tag{2.11}$$

and multiplication of (2.9) by $\sqrt{\sin\theta_m}$, we obtain

$$\sum_{k=1}^{N} \tilde{A}_{mk}\tilde{\psi}(\theta_k) = \mathscr{E}\tilde{\psi}(\theta_m), \tag{2.12}$$

where

$$\tilde{A}_{mk} = \left(\sqrt{\sin\theta_m/\sin\theta_k}\right) A_{mk} \tag{2.13}$$

is a symmetric matrix. Also, the wave function can be chosen to be an eigenfunction $\psi_\pm$ of a parity operation [8]

$$\psi_\pm(1-x) = \psi_\pm(\pi-\theta) = \pm\psi_\pm(\theta) = \pm\psi_\pm(x). \tag{2.14}$$

Equation (2.12) can then be converted to an equation with a matrix of dimensions reduced by roughly half.

B. *Gauss–Chebyshev*

Many singular integro-differential equations have been solved numerically by methods based on quadrature formulas [10]. Equation (1.5) falls into a class of

equations considered, and we can directly apply a method based on Gauss–Chebyshev quadrature. These methods have the advantage that the singular integral is accurately represented without the necessity of actually doing an integral, as was the case for the Multhopp method. Thus they are more generally applicable.

The Gauss–Chebyshev quadrature formula for a singular integral is [10]

$$\mathscr{P}\int_{-1}^{1}\frac{\sqrt{1-t'^2}}{t-t'}f(t')\,dt' \approx \sum_{k=1}^{N} A_k \frac{f(t_k)}{t-t_k} + \pi f(t)\frac{T_{N+1}(t)}{U_N(t)}, \tag{2.15}$$

where the A_k are weights given by

$$A_k = \frac{\pi}{N+1}\sin^2\frac{k\pi}{N+1}, \tag{2.16}$$

T_N and U_N are Chebyshev polynomials

$$T_N(t) = \cos[N\cos^{-1}t], \tag{2.17}$$

$$U_N(t) = \frac{\sin[(N+1)\cos^{-1}t]}{\sqrt{1-t^2}}, \tag{2.18}$$

and the t_k are the roots of U_N:

$$t_k = \cos\frac{k\pi}{N+1}. \tag{2.19}$$

This is a generalization of the ordinary Gauss–Chebyshev quadrature formula

$$\int_{-1}^{1}\sqrt{1-t'^2}\,f(t')\,dt' \approx \sum_{k=1}^{N} A_k f(t_k). \tag{2.20}$$

The weights A_k are the same.

We apply these formulas to the integrals in (1.5). A new integration variable $t = 2x-1$ is used, and a new function, consistent with the boundary conditions (1.4), is defined by

$$f(t) = \frac{1}{\sqrt{1-t^2}}\psi\left(\frac{t+1}{2}\right). \tag{2.21}$$

On use of (2.21), (2.20), and (2.15), Eq. (1.5) becomes, approximately,

$$\frac{8}{\pi}\frac{R-2}{\sqrt{1-t^2}}f(t) + \frac{2}{\pi}\sum_{k=1}^{N} A_k f(t_k) + \frac{8}{\pi}\frac{d}{dt}\left[\sum_{k=1}^{N} A_k \frac{f(t_k)}{t-t_k} + \pi f(t)\frac{T_{N+1}(t)}{U_N(t)}\right] = \mathscr{E}\sqrt{1-t^2}\,f(t). \tag{2.22}$$

An integration by parts has been used in the second integral to place the derivative outside the integral before the quadrature formula was applied. The derivative can now be carried out explicitly, except on f.

The choice of collocation points is again important. The term in (2.22) that contains the derivative of f can be eliminated by choosing as these points the roots of T_{N+1}, which are

$$z_j = \cos\left(\frac{2j-1}{2N+2}\pi\right), \qquad j = 1, \ldots, N+1. \tag{2.23}$$

If this choice is not used, one is forced to employ a finite-difference approximation for the derivative,thus introducing additional error.

We now have a linear system of $N+1$ equations; however, this is at a cost of $N+1$ additional unknowns $f(z_j)$, which brings the total of unknowns to $2N+1$. Rather than increase the number of equations, we redress this imbalance by reducing the number of unknowns via Lagrange interpolation:

$$f(t_k) = \sum_{j=1}^{N+1} L_j(t_k) f(z_j) \tag{2.24}$$

with

$$L_j(t) = \prod_{\substack{m=1 \\ m \neq j}}^{N+1} \frac{t - z_m}{z_j - z_m}. \tag{2.25}$$

The matrix eigenvalue problem obtained in this way is

$$\sum_{m=1}^{N+1} B_{jm} f(z_m) = \mathscr{E} f(z_j), \tag{2.26}$$

where

$$B_{jm} = \frac{1}{\sqrt{1-z_j^2}} \left\{ \delta_{jm} \left[\frac{8}{\pi} \frac{R-2}{\sqrt{1-z_j^2}} + 8 \frac{d}{dt} \left(\frac{T_{N+1}(t)}{U_N(t)} \right)_{t=z_j} \right] + \frac{8}{\pi} \sum_{k=1}^{N} A_k L_m(t_k) \left(\frac{1}{4} - \frac{1}{(z_j - t_k)^2} \right) \right\}. \tag{2.27}$$

On use of various identities [13], the remaining derivative can be expressed directly in terms of polynomials:

$$\frac{d}{dt}\left(\frac{T_{N+1}(t)}{U_N(t)}\right) = \frac{(N+1)[T_N(t) - tT_{N+1}(t)] - T_{N+1}(t)[U_{N-1}(t)/U_N(t) - Nt]}{(1-t^2)\, U_N(t)}. \tag{2.28}$$

As for the Multhopp method, the matrix eigenvalue problem (2.26) is solved by standard means.

The need for interpolation can be eliminated, as shown by Ioakimidis and Theocaris [10], by pairing the Gauss–Chebyshev method with one based on Lobatto–Chebyshev quadrature. The collocation and quadrature points are interchanged relative to the first method, and together they form the set of points where the solution is calculated.

C. *Finite Element*

At the heart of the finite-element method [11] is the subdivision of the domain of the unknown function into regions called elements. The unknown function is expanded in terms of basis functions defined on these regions. In the present case, the elements are segments of the unit interval. We choose N elements of equal length h.

The function ψ is to be calculated at N' points x_i called nodes, which are chosen at the endpoints of the elements and, perhaps, in the interior. A basis function ϕ_i is defined for each node such that

$$\phi_i(x_j) = \delta_{ij}. \tag{2.29}$$

The expansion

$$\psi(x) \approx \sum_{j=0}^{N'} \alpha_j \phi_j(x) \tag{2.30}$$

is then exact at the nodes, provided

$$\alpha_j = \psi(x_j). \tag{2.31}$$

We will always choose nodes in such a way that

$$x_0 = 0, \qquad x_{N'} = 1. \tag{2.32}$$

The boundary conditions (1.4) then imply that

$$\alpha_0 = 0, \qquad \alpha_{N'} = 0. \tag{2.33}$$

On substitution of (2.30) and use of (2.33), Eq. (1.5) becomes

$$\frac{2}{\pi}\frac{R-2}{x(1-x)} \sum_{j=1}^{N'-1} \alpha_j \phi_j(x) + \frac{4}{\pi} \int_0^1 dx' \sum_{j=1}^{N'-1} \alpha_j \phi_j(x') + \frac{4}{\pi} \mathscr{P} \int_0^1 \frac{dx'}{x-x'} \sum_{j=1}^{N'-1} \alpha_j \phi_j'(x') - \mathscr{E} \sum_{j=1}^{N'-1} \alpha_j \phi_j(x) = 0. \tag{2.34}$$

Rather than choose collocation points, we extract $N'-1$ linear equations from

(2.34) by applying the Galerkin method [11]. We require that the scalar product of ϕ_i with the left-hand side of (2.34) be zero for i between 1 and $N'-1$. This yields

$$\sum_{j=1}^{N'-1}\left[\frac{2}{\pi}(R-2)\,I_{ij}^{(1)}+\frac{4}{\pi}I_{ij}^{(2)}+\frac{4}{\pi}I_{ij}^{(3)}-\mathscr{E}I_{ij}^{(4)}\right]\alpha_j=0, \tag{2.35}$$

where

$$I_{ij}^{(1)}=\int_0^1 dx\,\frac{\phi_i(x)\,\phi_j(x)}{x(1-x)}, \tag{2.36}$$

$$I_{ij}^{(2)}=\int_0^1 dx\,\phi_i(x)\int_0^1 dx'\,\phi_j(x'), \tag{2.37}$$

$$I_{ij}^{(3)}=\int_0^1 dx\,\mathscr{P}\int_0^1 dx'\,\frac{\phi_i(x)\,\phi_j'(x')}{x-x'}, \tag{2.38}$$

$$I_{ij}^{(4)}=\int_0^1 dx\,\phi_i(x)\,\phi_j(x). \tag{2.39}$$

We thus arrive at a generalized eigenvalue problem

$$\sum_{j=1}^{N'-1}C_{ij}\psi(x_j)=\mathscr{E}\sum_{j=1}^{N'-1}D_{ij}\psi(x_j) \tag{2.40}$$

with

$$C_{ij}=\frac{2}{\pi}(R-2)\,I_{ij}^{(1)}+\frac{4}{\pi}I_{ij}^{(2)}+\frac{4}{\pi}I_{ij}^{(3)} \tag{2.41}$$

and

$$D_{ij}=I_{ij}^{(4)}. \tag{2.42}$$

Given explicit expressions for the matrices, the matrix problem can, again, be solved by standard methods.

To make the form of the matrices explicit, we must choose nodes and basis functions. The simplest choice is an assignment of nodes only at the endpoints of the element, so that $x_i=ih$, combined with basis functions of the form

$$\phi_i(x)=\begin{cases}1-|x-x_i|/h, & |x-x_i|\leqslant h,\\ 0, & \text{otherwise.}\end{cases} \tag{2.43}$$

The elements are then called linear elements.

The next simplest choice is known as a quadratic element. Nodes are placed at

the endpoints and the midpoint, with $x_i = ih/2$. The basis function for a midpoint node is

$$\phi_i(x) = \begin{cases} (4/h^2)(x_{i+1} - x)(x - x_{i-1}), & |x - x_i| \leqslant h/2, \\ 0, & \text{otherwise}, \end{cases} \tag{2.44}$$

and for an endpoint node,

$$\phi_i(x) = \begin{cases} (2/h^2)(x - x_{i-1})(x - x_{i-2}), & x_i > x > x_i - h, \\ (2/h^2)(x_{i+1} - x)(x_{i+2} - x), & x_i < x < x_i + h, \\ 0, & \text{otherwise}. \end{cases} \tag{2.45}$$

In either case, calculation of the matrices $I_{ij}^{(k)}$ is straightforward. Because the results are somewhat lengthy, but not informative, we do not report them here.

III. Discussion

A comparison of results from the different methods is given in Fig. 1. Each matrix eigenvalue converges toward the exact answer as the dimension n of the matrix is increased, but some converge more rapidly than others. The rate depends on the value of R and on the method used. For large R, which corresponds to small coupling, convergence is much faster than for small R. This is not surprising since for zero coupling the solution is trivial.

The best individual method, for small and moderate R, is Gauss–Chebyshev. The Multhopp method is nearly as good. For large R, all methods do well except when the matrix is very small.

The error in the result from any one method, with $n = 25$, ranges from 10% for small R to 1% for large R. Use of extrapolation techniques could considerably reduce the error for small R.

Although the Gauss–Chebyshev and Multhopp methods work well for the one-pair equation (1.5), they are not applicable to the coupled set of one-pair and two-pair equations [7]. The asymmetric Gauss–Chebyshev matrix does a poor job of modeling the Hermitian system. In the case of the Multhopp method, the matrix problem cannot even be formulated because some of the integrals analogous to (2.4) cannot be done.

Of the methods discussed here, the one best suited for the two-pair equations is the quadratic finite-element method. It is equivalent to the use of linear square elements in the 2-dimensional space of x_1 and x_2. Therefore, an even-handed treatment of variables is possible. Extension to the four variables present in the two-pair case is direct. The elements then become hypercubes. The integrals involved in calculating the matrices can all be done, and the matrices are automatically symmetric. Work on this approach is in progress.

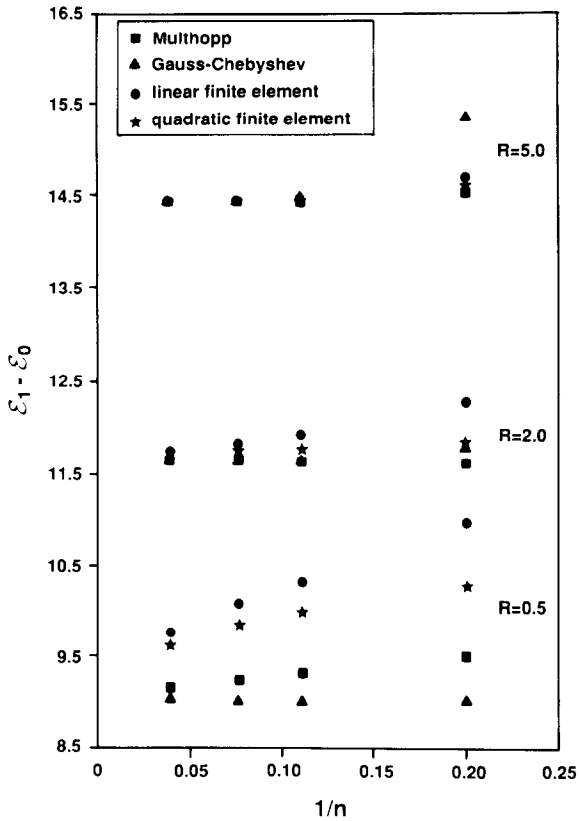

FIG. 1. The difference between the two lowest eigenvalues, $\mathcal{E}_0$ and $\mathcal{E}_1$, of matrix approximations to Eq. (1.5) as a function of the reciprocal of the matrix dimension n. Results are shown for the four different methods and three different values of R.

TABLE I

Eigenvalues of Eq. (1.5) for the Ground State and First Excited State

	Ground state		First excited state	
R	Eller *et al.*	Quadratic finite element	Eller *et al.*	Quadratic finite element
0.098	2.912	3.297	10.99	12.01
0.393	4.276	4.520	13.14	13.82
1.571	8.876	8.928	19.89	20.08
6.283	23.94	23.94	39.16	39.22
25.13	77.48	77.24	99.81	99.74
100.5	276.7	278.1	310.9	312.6

Note. The results of the quadratic finite-element method are compared with those of Eller, Pauli, and Brodsky [9] at various values of R. Both methods approximate (1.5) by a matrix eigenvalue problem, either directly or indirectly. The matrix dimension used in 49 for the quadratic finite-element method and 240 for the method of Ref. [9].

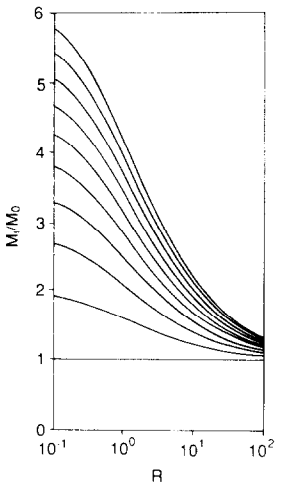

FIG. 2. The spectrum as a function of R. The ten lowest mass states are represented in terms of ratios M_i/M_0 to the ground-state mass M_0. These were calculated using the quadratic finite-element method, with a matrix dimension of 49, and the definition of $\mathscr{E}$ in (1.6), which yields $M_i/M_0 = \sqrt{\mathscr{E}_i/\mathscr{E}_0}$.

Table I allows comparison of the quadratic finite-element method with the discretized light-cone quantization method of Ref. [9]. The table lists eigenvalues of the one-pair equation (1.5). The relatively large differences at small R are to be expected because differently sized matrices were used.

In Fig. 2 we present the spectrum as a function of R. The calculations were done with the quadratic finite-element method. The trend in the relative magnitudes of the excitation energies is easily explained in terms of the definition of R in (1.6). Stronger coupling should produce larger excitations, and that is just what it seen as R is reduced.

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REFERENCES

1. J. SCHWINGER, *Phys. Rev.* **128**, 2425 (1962).
2. See, for example, C. G. CALLAN, N. COOTE, AND D. J. GROSS, *Phys. Rev. D* **13**, 1649 (1976).
3. We do not consider lattice gauge theories.
4. P. A. M. DIRAC, *Rev. Mod. Phys.* **21**, 392 (1949).
5. For reviews, see S. J. BRODSKY, T. HUANG, AND G. P. LEPAGE, "Hadronic and Nuclear Interactions in QCD," *Quarks and Nuclear Forces*, edited by D. C. Fries and B. Zeitnitz (Springer-Verlag, Berlin, 1982); G. P. LEPAGE, S. J. BRODSKY, P. B. MACKENZIE, AND T. HUANG, "Hadronic Wave Functions in QCD," *Banff Summer Institute on Particle Physics, Banff, Alberta, Canada, August 1981.*

6. G. 'T HOOFT, *Nucl. Phys. B* **75**, 461 (1974); M. S. MARINOV, A. M. PERELOMOV, AND M. V. TERENT'EV, *JETP Lett.* **20**, 225 (1974); C. G. CALLAN, N. COOTE, AND D. J. GROSS, Ref. [2].
7. Y. MA, M.S. thesis, University of Minnesota, Duluth, 1986 (unpublished).
8. A. J. HANSON, R. D. PECCEI, AND M. K. PRASAD, *Nucl. Phys. B* **121**, 477 (1977).
9. T. ELLER, H.-C. PAULI, AND S. J. BRODSKY, *Phys. Rev. D* **35**, 1493 (1987).
10. F. ERDOGAN AND G. D. GUPTA, *Q. Appl. Math.* **29**, 525 (1972); M. M. CHAWLA AND T. R. RAMAKRISHNAN, *BIT* **14**, 14 (74); P. S. THEOCARIS, *Serdica* (*Bulgaria*) **2**, 252 (1976); N. I. IOAKIMIDIS AND P. S. THEOCARIS, *Q. Appl. Math.* **37**, 325 (1979); P. S. THEOCARIS AND G. TSAMASPHYROS, *Appl. Math. Comput.* **15**, 47 (1984).
11. For an introduction to finite-element methods see, for example, R. VICHNEVETSKY, *Computer Methods for Partial Differential Equations*, Vol. I (Prentice–Hall, Englewood Cliffs, NJ, 1981).
12. H. BERGKNOFF, *Nucl. Phys. B* **122**, 215 (1977).
13. M. ABRAMOWITZ AND I. A. STEGUN, *Handbook of Mathematical Functions* (Dover, New York, 1972), p. 783.